Naive Infinity Categories

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Until now I have not really understood what was the "idea" of infinity categories was, and how it was related to the formalism. I think Im getting a handle on it so here goes. The idea is weakening composition, in a coherent way.

The first thing to notice is that many structures do not *naturally* form one categories. This can happen in two senses, one is that the one category does not capture the right notion of equivalence, the other is when functions do not strictly associate.

Example (Associativity)**.** *Consider the category of sets. Then we can look at spans in this category, that is sets, where a map between two sets A, C is a "span" or a "roof" $A \leftarrow B \rightarrow C$. These come up as the diagrams from which we construct pullbacks and pushouts etc. Composition we want to define as pulling back.*

$$\begin{tikzcd}
 & & F \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & C \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr] & \\
A & & B & & D
\end{tikzcd}$$

The outer roof is a map from $A \rightarrow D$. But consider compositions of three such roofs

$$\begin{tikzcd}
 & & & D_1 \arrow[dl, dotted] \arrow[ddrrr, dotted] & & & \\
 & & C_1 \arrow[dl, dashed] \arrow[dr, dashed] & & & & \\
 & B_1 \arrow[dl] \arrow[dr] & & B_2 \arrow[dl] \arrow[dr] & & B_3 \arrow[dl] \arrow[dr] & \\
A_1 & & A_2 & & A_3 & & A_4
\end{tikzcd}
\overset{?}{=}
\begin{tikzcd}
 & & & D_2 \arrow[ddll, dotted] \arrow[dr, dotted] & & & \\
 & & & & C_2 \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & B_1 \arrow[dl] \arrow[dr] & & B_2 \arrow[dl] \arrow[dr] & & B_3 \arrow[dl] \arrow[dr] & \\
A_1 & & A_2 & & A_3 & & A_4
\end{tikzcd}$$

Clearly there is no reason that they will be the same and so this does not form a category because it is not associative. One can make it a weak two category by saying that two morphisms are again spans such that the following diagram commutes

$$\begin{tikzcd}
 & A & \\
C \arrow[ur] \arrow[dr] & D \arrow[l] \arrow[r] & C' \arrow[ul] \arrow[dl] \\
 & B &
\end{tikzcd}$$

Example (Equivalence)**.** *The obvious example is that of the category of categories themselves. Although functor strictly associate, we typically do not care about categories up to isomorphism, but only up to equivalence.*

Now both of these things can be patched up, in sort of ad hoc ways. We can manually add inverses to fix the isomorphisms, or we can manually mod out by some isomorphism classes to fix the associativity, however there is not a principled reason why we shouldnt be able to consider these structures for what they are. But once you start yo might as well go all the way.

What is an infinity category then. It is the structure that thakes these two notions seriously. First we consider a set of objects. Then between each pair of objects there should be a set of maps. Now usually we would require some axioms on these maps, however now we will loosen these axioms. In particular between any two maps we will associate another set of maps, and so on ad infinitum. We require some restrictions, namely the one morphisms should associate, but only *up to isomorphism*, where an isomorphism is itself an inverse *only up to higher maps*. Again the two morphisms should associate up to isomorphism where both of these concepts are determined by the three morphisms, and so on ad infinitum. This is all a naive infinity category is, its just a category but instead of stictly associating and composing we consider them only up to isomorphism in some sort of recursive way. In particular it would make sense to say things like "let blah be an object of my infinity category", "let bleh be some one morphisms".

There are many instantiations of this idea in ordinary one categories, that are in whatever sense equivalent. But this is the idea, and we shouldnt loose sense of the ways in which those concrete models are trying to capture this idea.

1 The Infinity Category of Spaces

With this naive model in mind what would an infinity category of (topological) spaces be, lets denote is $\mathcal{S}$. The objects should be spaces. The morphisms would be topological spaces, the two morphisms homotopies between continuous maps, the three morphisms homotopies between homotopies etc etc.

What would an isomorphism in this category. Well we should take two objects $x, y \in \mathcal{S}$ and a morphism between them $f \in \mathrm{Hom}(x, y)$, or $x \to y$. Ordinarily we would require a continuous inverse $g : y \to x$ such that $gf = \mathrm{id}_x, fg = \mathrm{id}_y$. However our philosophy is that we shouldnt care about equality but isomorphism up to two morphism. Hence we should look for a continuous inverse *and* a homotopies between the compositions and the respective identities. So a g and two homotopies such that $fg \simeq \mathrm{id}, gf \simeq \mathrm{id}$. Notice that we havent simply "modded out" by the higher homotopies, saying well two maps are equivilant if there exists some homtopy, we actually leave the structure all in place and pick homotopies to the identity. Really explicitly we are looking at the objects $\mathrm{id}_x, gf \in \mathrm{Hom}(x, x), \mathrm{id}_y, fg \in \mathrm{Hom}(y, y)$ and asking for there to be a two *isomorphism* between them. Notice that by definition of homotopy equivalence just the existence of *any* two morphism guarantees that the two spaces are homotopy equivalent. What we ask for is however a priori stronger, as we want not just a two morphism but one that is invertible (up to three morphisms etc). The discrepency is cleared up when one notices that being homotopic is an equivalence relation on morphisms and in particular every homotopy is invertible.

At this point it makes sense to introduce (n, m)-categories. That is an n-category such that all morphisms $> m$ are invertible. What we have so far been talking about would be an (∞, ∞)-category. Given such a category we can just throw out all the higher morphisms that are not invertible to get for instance an $(\infty, 1)$-category, where only the one morphisms can be non-invertible. Our observation amounts to showing that the (∞, ∞)-category of spaces is *already* the $(\infty, 1)$-category of spaces. Notice that in $(\infty, 1)$-category an isomorphism is a one morphisms that is isomorphic to the identity, that is has an inverse up to three morphisms, however now because all two morphisms are such isomorphisms (in the higher sense) we can simply reduce this to asking that there be some two morphism in the $(\infty, 1)$-category. So if an $(\infty, 1)$-category comes from an (∞, ∞)-category then the notion of isomorphism of objects should be the same?

Thus it is clear that reasoning about infinity categories naively we have the following

Lemma (Naive)**.** *Two spaces are homotopy equivalent iff they are isomorphic in $\mathcal{S}$.*

We want more to be true. We would like to formulate all our category theory so that it just works in this naive world. That is limits colimits, functors etc should all behave how we expect them to (or better). Some things that we *want* to be true of the (infinity) category of spaces would be that it is a topos (think of all of Steenrods hard work), that limits and colimits are homotopy coherent, in the sense that they provide the LES in homotopy groups that we want, etc.

Lemma (Naive)**.** *Let $f : X \to Y$ be a continuous map, then its homotopy limit (in the sense of Quillen) is homotopy equivalent to the infinity categorical limit of $f : X \to Y$ when thought of as a one morphism in $\mathcal{S}$.*

Lemma (Dream)**.** *$\mathcal{S}$ is an (∞) topos.*

2 Being Less Naive

I know that this is the naive document, but lets look at the quasi-category instantiation of infinity categories here, see how it tracks our naive ideas and how our theorems are instantiated. The key ingredients are the two pairs of adjoints [Lan, Prop 1.1.36, 1.2.18]

$$\begin{tikzcd}
\mathrm{Top} \arrow[dr, bend left=10, "\mathrm{Sing}"'] \arrow[dr, phantom, "\perp"] & \\
\pitchfork & (\mathrm{large}) - \mathrm{sSet} \arrow[ul, bend right, "|-|"'] \arrow[dl, bend left, "h"] \\
(\mathrm{large}) - \mathrm{Cat} \arrow[ur, bend right=10, "N"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Recall that in the non-naive world a quasi-category, $(\infty, 1)$-category or simply ∞-category are identified and by definition are certain types of simplicial sets. If we want an infinity category of spaces then it means a simplicial set of spaces, or in particular a simplicial set whose *zero simplicies* are spaces. The one simplicies are the mapping classes etc etc. One way to make this really precise is to simplicially enrich Top and then apply the coherent nerve [Lan, Def 1.2.63]. To some extent the notion of a simplicial category is closer to our naive definition, and the coherent nerve is just a way of going between two different models of infinity categories.

Thus when we want to define categorical things, they have to be *inside the simplicial set.* So we would set up a diagram of zero and one simplicies and then try to pull it back etc. [Lan, Def 1.3.47] defines for two "objects" of an infinity category its mapping space, that is for two 0-simplicies a set of maps. Now there it is again defined as a simplicial set. This makes sense if we think naively, the maps between any two objects should themselves have higher maps between them etc. So we can see that in an infinity category any mapping space must itself be an infinity category. If we are in the (∞, n)-formalism then taking the mapping class should just shift n down one.

Lemma. *Two spaces are weakly homotopy equivalent iff they are isomorphic in $\mathcal{S}$.*

Proof. Now $\mathcal{S}$ is the coherent nerve of the simplicially enriched category of topological spaces. This is an infinity category and we are trying to show that if two of its zero simplicies are isomorphic then, because those zero simplicies happen to be spaces, they are also homotopy equivalent.

h sends this simplicial set of all topological spaces to the homotopy category associated which in this case is just the homotopy category of Top [Yan, Thm 2.2.11] (in the sense of Quillen and hence weak homotopy). Now h is a functor, so it preserves equivalences, however we are not looking at equivalences of simplicial sets, which would just give us alternative presentations of the homotopy category of spaces. What we need is to look inside $h\mathcal{S}$. The fact is that h *also* sends equivalences in S to equivalences in hS [Lan, Observation 1.2.8], which *does not follow from functoriality* but from the construction.

The converse is more or less clear from the coherent nerve construction and the definition of the enrichment.

We have really used here also the fact that the infinity category of spaces has two equivalent (as infinity categories) presentations, one as the coherent nerve of the category of topological spaces, the other as the coherent nerve of the category of infinity groupoids [Lur09, Remark 1.2.16.3][Yan, Thm 3.3.3]. Nonte that here we are also just mapping a simplicial set, the infinity category of spaces to a normal category. This shouldnt be confused with a functor that maps from the *infinity category* of infinity categories. We have no use for it here, our maps are just on the zero simplicies of this infinity category of infinity categories if you like. On the other hand we are using that $N_\Delta \mathrm{Top}_\bullet \simeq N_\Delta \mathrm{Kan}$ which is a statement in this infinity category of infinity categories. In particular we want to know that

Lemma (Kerodin 01E3)**.** *If two infinity categories are equivalent as infinity categorie they have the same homotopy category. The converse does not hold.*

Note that there he says that an equivalence of infinity categories gives a homotopy equivalence of the simplicial sets, but this is by definition for him an isomorphism of the homotopy categories of these simplicial sets. Note that it is *strictly weaker* to be equivalent in the infinity category of infinity categories than being equivalent in the one category of simplicial sets. That natural isomorphism implies infinity equivalence is clear, the fact that the converse fails we have been told can be wittnessed by EG for some group G which is contractable in the infinity categorical sense, but not as a simplicial set. So we have the implications for simplicial sets which are moreover all strict:

$$\text{naturally isomorphic} \Longrightarrow \text{equiv as infinity cats} \Longrightarrow \text{homotopy equivalent} \ .$$

Comparing to the situation for simplicial sets we have the similar diagram for spaces

$$\text{homeomorphic} \Longrightarrow \text{equiv in } \mathcal{S} \Longleftrightarrow \text{homotopy equivalent} \ .$$

What we can see here is the difference between infinity groupoids (spaces) and general infinity categories.

The fact that the infinity category of spaces is an infinity topos is almost tautological from the point of view of infinity categories. Finally we have that our notion of homotopy pullbacks from Quillen really are made more uniform by the theory

Lemma ([Lur09], §4.2.4)**.** *Let F be an (ordinary) functor between two (simplicially enriched) categories, if the categories are fibrant then then an object (of the domain category) is a homotopy colimit of the diagram (in the sense of Quillen) iff the coherent nerve of the colimit is an is an infinity categorical colimit.*

Remark. Given a simplicially enriched category we can do two things, the first apply the coherent nerve, the second forget its simplicial enrichment to get a normal category and then take the ordinary nerve. These two simplicial sets seem to have no relationship, neither one categorically nor infinity categorically. I cannot make this precise (exhibit examples) at this time.

Remark. An example of applying this theory in a somewhat elementary way is given here for the Freudenthal suspension theorem.

References

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